Pre-class Warm-up!!!

Find the best way to complete the sentence:

One Joule is

a. the force required to accelerate 1 gram at 1 cm/sec

b. the energy required to raise the temperature of 1 cubic centimeter of water by 1 degree centigrade

c. the work done in lifting 1 kilogram through 1 meter

d. the work done by 1 Newton moving a distance of 1 meter

e. None of the above. Maybe a type of candy.

Does Joule have a capital J

Yes

NO

7.1 The path integral 7.2 The line integral

We learn:

- Two closely related integrals along a path or curve
- How to set them up and evaluate them
- Physical interpretation
- Different notation for these integrals
- Orientation of a curve
- Theoretical things: independence of the parametrization of the curve (up to orientation for line integrals); Riemann sums.
- The line integral of a gradient vector field

We do not need to know:

- Total curvature in 7.1
- Simple curves, closed curves in 7.2
- You are not tested on the 'theoretical things'.

Types of question:

- mainly evaluate the integrals

The two kinds of integral

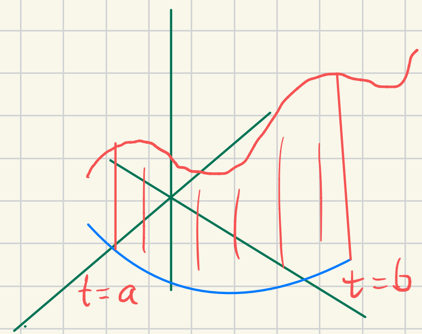
Both take a path c : [a,b] -> R^n
(In the book n = 3 .)
We want to assume $c'(t) \neq 0$ always.
Thus c traces along the curve from one end to the other, without retracing itself.

The orientation of c is the direction in which we travel along the curve.
There are two possible orientations.

For the path integral in 7.1 we take a scalar function f : R^n -> R and define

$$\int_c f\, ds = \int_a^b f(c(t))\, \|c'(t)\|\, dt.$$

If $f(c(t)) = 1$ for all t we get the length of the curve between $t=a$ and $t=b$. Generally, we get the curvy area under the graph of f.



For the line integral in 7.2 we take a vector field F : R^n -> R^n and define

$$\int_c F \cdot d\underline{s} = \int_a^b F(c(t)) \cdot c'(t)\, dt$$

Use for calculating

$$= \int_c F_1\, dx_1 + F_2\, dx_2 + \cdots + F_n\, dx_n$$

The last and first integrals are notation.

Think $dx_i = \frac{dc_i}{dt} \cdot dt = c_i' dt$

and $d\underline{s} = c'(t)\, dt$.

Physical interpretation : Work done by the force field moving from $t=a$ to $t=b$.

Questions in 7.1

These are either: find a parametrization of a given curve. Sometimes it is given in pieces and you have to parametrize it in pieces.
Or: evaluate the integral.

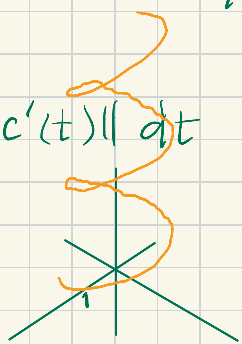
Bad question: this could be negative

Like question 10.
A wire is bent into a helix parametrized by $c(t) = (\cos t, \sin t, t)$ where $0 \le t \le \pi$. The wire has variable line-density that is $\boxed{xy + z}$ at position (x,y,z). Find the mass of the wire.

Line density could be measured in g/cm

The mass is

$$\int_c (xy+z)\,ds = \int_0^{\pi} (xy+z)\,\|c'(t)\|\,dt$$



$$= \int_0^{\pi} (\cos t \sin t + t)\sqrt{\sin^2 t + \cos^2 t + 1^2}\,dt$$

$$= \left[\left(\frac{1}{2}\sin^2 t + \frac{t^2}{2}\right)\sqrt{2}\right]_0^{\pi}$$

$$= \left(0 + \frac{\pi^2}{2} - 0 - 0\right)\sqrt{2} = \frac{\pi^2\sqrt{2}}{2}$$

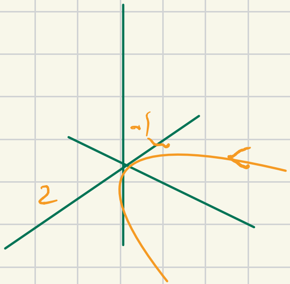
Physical interpretation:

Area under the curvy graph.

Questions in 7.2

These are: calculate the integral

Like Question 5:
Calculate the work done by the force field $F(x,y,z) = (y,-x,z)$ in moving a particle along the parabola $y = x^2$, $z = 0$ from $x = -1$ to $x = 2$.



Step 1: parametrize the parabola.

$c(t) = (t, t^2, 0) \quad -1 \le t \le 2$

$c'(t) = (1, 2t, 0)$

The work done is $\int_{-1}^{2} (y,-x,z)\cdot(1,2t,0)\,dt$

$= \int_{-1}^{2} (t^2,-t,0)\cdot(1,2t,0)\,dt$

$= \int_{-1}^{2} -t^2\,dt = \left[-\frac{t^3}{3}\right]_{-1}^{2}$

$= \frac{-8}{3} - \frac{1}{3} = -3$

Question: What does it mean if the work done by a force field is negative?

a. the question was wrong
b. the method of doing the question was wrong
c. energy was transferred from the particle ✓
d. energy was transferred to the particle
e. None of the above.

Theorems 1 and 2 of 7.2

Suppose c and p are two parametrizations of the same curve, with the same orientation. Then

$$\int_c F \cdot d\underline{s} = \int_p F \cdot d\underline{s}$$

If c and p have the opposite orientation then

$$\int_c F \cdot d\underline{s} = -\int_p F \cdot d\underline{s}$$

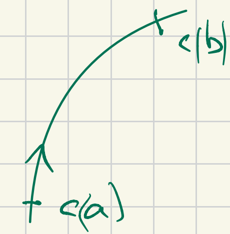
In either case

$$\int_c f\, ds = \int_p f\, ds$$

Theorem 3 of 7.2

Let f : R^n -> R and let c : [a,b] -> R^n be a path.

Then $\int_c \nabla f \cdot d\underline{s}$ =f(c(b)) - f(c(a)) .



Proof.

$$\int_c \nabla f \cdot d\underline{s} =$$

$$= \int_a^b \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right) \cdot \left(c_1'(t), \dots, c_n'(t)\right) dt$$

$$= \int_a^b Df(c(t)) \circ c'(t)\, dt = \int_a^b \frac{d}{dt} f(c(t))\, dt$$

$$= f(c(b)) - f(c(a)) \quad \square$$

Example

Find the work done by F(x,y) = (2xy, x^2) in moving along the path
c(t) = (√(1+t^2) , e^(sin t)) from t = 0 to t = 1.

Solution $F(x,y) = \nabla f$ where

$f(x,y) = x^2 y$ ~~so~~

$$\int_c F \cdot ds = f(c(1)) - f(c(0))$$

$$= f(\sqrt{2}, e^{\sin(1)}) - f(1,1)$$

$$= 2e^{\sin(1)} - 1 \quad \square$$